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Some Bonnesen-style Minkowski inequalities

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Full list of author information is available at the end of the article**Abstract**

In this paper, we obtain some Bonnesen-style Minkowski inequalities of mixed volumes of convex bodies K and L in the Euclidean space \mathbb{R}^n . Let L be the unit ball; we get some better Bonnesen-style isoperimetric inequalities than Dinghas's result for $n \geq 3$.

MSC: 52A20; 52A40**Keywords:** Minkowski inequality; Aleksandrov-Fenchel inequality; Bonnesen-style inequality; mixed volumes; isoperimetric inequality

1 Introduction

It is well known that the ball has the maximum volume among bodies of fixed surface area in the Euclidean space \mathbb{R}^n . That is, of all domains K with surface area $S(K)$ and volume $V(K)$ (cf. [1, 2]),

$$S(K)^n - n^n \omega_n V(K)^{n-1} \geq 0, \quad (1)$$

with equality if and only if K is a ball. Here ω_n denotes the volume of the unit ball,

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function.

The isoperimetric deficit

$$\Delta_n(K) = S(K)^n - n^n \omega_n V(K)^{n-1} \quad (2)$$

measures the deficit between the domain K and a ball of radius $(S(K)/n\omega_n)^{1/(n-1)}$. A Bonnesen-style isoperimetric inequality is of the form (cf. [2–4])

$$\Delta_n(K) = S(K)^n - n^n \omega_n V(K)^{n-1} \geq B_K, \quad (3)$$

where the quantity B_K is a non-negative invariant of geometric significance of K and vanishes only when K is a ball.

Bonnesen himself proved several inequalities of the form (3) in the Euclidean plane (cf. [5, 6]), but he was not able to obtain direct generalizations of his two-dimensional results. This was done much later, first by Hadwiger [7] for $n = 3$, and then by Dinghas [8]

for arbitrary dimension. From then on, some Bonnesen-style inequalities in the higher dimensions and generalizations have been obtained by Osserman (*cf.* [1, 2]), Santaló (*cf.* [9]), Groemer and Schneider (*cf.* [10]), Zhang (*cf.* [11]), Zhou (*cf.* [4, 12]) and others. See references [13–36] for more details. The following well-known Bonnesen-style inequality for a convex body K in the Euclidean space \mathbb{R}^n is due to Dinghas (*cf.* [8]):

$$S(K)^n - n^n \omega_n V(K)^{n-1} \geq (S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)} r)^{n(n-1)}, \quad (4)$$

where r is the in-radius of K , and equality holds if and only if K is a ball.

In [11], some different forms of Bonnesen-style isoperimetric inequalities have been established associated with the mean width of K . Zhang obtained (*cf.* [11])

$$\left(\frac{M(K)}{2}\right)^{n/(n-1)} - \left(\frac{V(K)}{\omega_n}\right)^{1/(n-1)} \geq \left(\frac{V(K)}{\omega_n}\right)^{n/(n-1)} \left(\left(\frac{V(K)}{\omega_n}\right)^{-1/n} - R^{-1}\right),$$

where $M(K)$ and R are the mean width and out-radius of K , respectively.

The Minkowski inequality of mixed volume is a natural generalization of the isoperimetric inequality (1) in the Euclidean space \mathbb{R}^n (*cf.* [27, 37–39]). Let K, L be convex bodies in \mathbb{R}^n , then

$$V_1(K, L)^n \geq V(K)^{n-1} V(L), \quad (5)$$

where $V_1(K, L)$ is the mixed volume of K and L and the equality holds if and only if K and L are homothetic.

Motivated by (2), we define the Minkowski homothetic deficit as

$$\Delta_n(K, L) = V_1(K, L)^n - V(K)^{n-1} V(L). \quad (6)$$

The Minkowski homothetic deficit $\Delta_n(K, L)$ measures the homothety between K and L . Then a Bonnesen-style Minkowski inequality would be of the form

$$\Delta_n(K, L) = V_1(K, L)^n - V(K)^{n-1} V(L) \geq B_{K,L}, \quad (7)$$

where the quantity $B_{K,L}$ is an invariant of geometric significance about K and L with the following basic properties:

1. $B_{K,L}$ is non-negative;
2. $B_{K,L}$ vanishes only when K and L are homothetic.

Note that let L be the unit ball B and by $S(K) = nV_1(K, B)$, the surface area of K , then the Minkowski homothetic deficit is just the isoperimetric deficit. Therefore, the Bonnesen-style Minkowski inequality (7) is more general than the Bonnesen-style isoperimetric inequality (3).

In this paper, we focus on Bonnesen-style Minkowski inequalities of type (7). Some $B_{K,L}$ are obtained. Let L be the unit ball; then we obtain stronger Bonnesen-style isoperimetric inequalities K than (4).

2 Preliminaries

A set of points K in the Euclidean space \mathbb{R}^n is *convex* if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $\lambda x + (1 - \lambda)y \in K$. A *domain* is a set with nonempty interiors. A *convex body* is a compact convex domain. The set of convex bodies in \mathbb{R}^n is denoted by \mathcal{K}^n . Let \mathcal{K}_o^n be the class of members of \mathcal{K}^n containing the origin in their interiors. Write V for an n -dimensional Lebesgue measure and \mathcal{H}^{n-1} for an $(n - 1)$ -dimensional Hausdorff measure. S^{n-1} denotes the surface of the unit ball in \mathbb{R}^n .

A convex body $K \subset \mathbb{R}^n$ is uniquely determined by its support function $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$, where $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. For the support function of the dilate $cK = \{cx : x \in K\}$ of a convex body K we have

$$h_{cK} = ch_K, \quad c > 0. \quad (8)$$

Note that support functions are positively homogeneous of degree one and subadditive. It follows immediately from the definition of support functions that for convex bodies K and L

$$K \subseteq L \iff h_K \leq h_L. \quad (9)$$

For a convex body K and each Borel set $\omega \subset S^{n-1}$, the reverse spherical image $\tau(K, \omega)$, of K at ω is the set of all boundary points of K which have an outer unit normal belonging to the set ω . Associated with each convex body $K \in \mathcal{K}_o^n$ there is a Borel measure S_K on S^{n-1} called the Aleksandrov-Fenchel surface area measure of K , defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\tau(K, \omega)),$$

for each Borel set $\omega \subseteq S^{n-1}$. Observe that for the surface area measure of the dilate cK of K we have

$$S_{cK} = c^{n-1}S_K, \quad c > 0.$$

The Minkowski sum of convex sets K_1, \dots, K_m in \mathbb{R}^n is defined by

$$K_1 + \dots + K_m = \{x_1 + \dots + x_m : x_1 \in K_1, \dots, x_m \in K_m\}.$$

The mixed volume $V(K_1, \dots, K_n)$ of compact convex sets K_1, \dots, K_n in \mathbb{R}^n is defined by

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{j=1}^n (-1)^{n+j} \sum_{i_1 < \dots < i_k} V(K_{i_1} + \dots + K_{i_k}).$$

The Aleksandrov-Fenchel inequality about the i th mixed volume is

$$V_i(K_1, K_2)^2 \geq V_{i+1}(K_1, K_2)V_{i-1}(K_1, K_2), \quad (10)$$

where

$$V_i(K_1, K_2) = V(\underbrace{K_1, \dots, K_1}_{n-i}, \underbrace{K_2, \dots, K_2}_i)$$

with K_1 appears $n - i$ times and K_2 appears i times and (10) holds as an equality if and only if K and L are homothetic.

Note that

$$V_n(K_1, K_2) = V(K_2), \quad V_0(K_1, K_2) = V(K_1). \quad (11)$$

The following inequality for mixed volumes is the general Aleksandrov-Fenchel inequality: Let $K_1, \dots, K_n \in \mathcal{K}$ and $1 \leq m \leq n$. Then

$$V(K_1, \dots, K_n)^m \geq \prod_{i=1}^m V(K_i, \dots, K_i, K_{m+1}, \dots, K_n).$$

Hence

$$V_1(K_1, K_2)^{n-1} \geq V(K_1)^{n-2} V_{n-1}(K_1, K_2). \quad (12)$$

Let $K_2 = B$, then $V_i(K_1, B) = W_i(K_1)$, the i th quermassintegral of the convex body K_1 .

The mixed volume has monotonicity: If $K_1 \subset K'_1$, then

$$V(K_1, K_2, \dots, K_n) \leq V(K'_1, K_2, \dots, K_n).$$

The mixed volume $V_1(K, L)$ of the convex bodies $K, L \in \mathcal{K}_o^n$ has the integral form

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L dS_K. \quad (13)$$

Since

$$V(K) = V_1(K, K),$$

we have

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_K.$$

If B is the unit ball, then

$$nV_1(K, B) = S(K),$$

the surface area of K . The mean width $M(K)$ of K is

$$M(K) = \frac{2}{\omega_n} V_1(B, K),$$

that is,

$$M(K) = \frac{2}{n\omega_n} \int_{S^{n-1}} h_K dS_K.$$

The in-radius $r(K, L)$, out-radius $R(K, L)$ of K with respect to L are, respectively, defined by

$$r(K, L) = \sup\{\lambda : x \in \mathbb{R}^n \text{ and } x + \lambda L \subset K\},$$

$$R(K, L) = \inf\{\lambda : x \in \mathbb{R}^n \text{ and } K \subset x + \lambda L\}.$$

Notice that always

$$r(K, L)R(L, K) = 1.$$

When L is the unit ball, $r(K, L)$ and $R(K, L)$ are the radius of maximal inscribed and minimal circumscribed balls of K , respectively.

3 Bonnesen-style Minkowski inequalities associated with $r(K, L)$

In this section, we derive some Bonnesen-style Minkowski inequalities associated with in-radius $r(K, L)$ of K with respect to L . In [26], Diskant improved the Minkowski inequality of mixed volumes as follows.

Lemma 1 *Let K, L be convex bodies in the Euclidean space \mathbb{R}^n , then*

$$V_1(K, L)^{n/(n-1)} - V(K)V(L)^{1/(n-1)} \geq (V_1(K, L)^{1/(n-1)} - r(K, L)V(L)^{1/(n-1)})^n, \quad (14)$$

with equality if and only if K is homothetic to L .

Note that the right-hand side of (14) is non-negative for $x + r(K, L)L \subseteq K$ ($x \in \mathbb{R}^n$). By (13) we have

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L dS_K \geq \frac{1}{n} \int_{S^{n-1}} h_L dS_{r(K, L)L} \geq r(K, L)^{n-1} V(L).$$

From Lemma 1 and using the inequality $x^{n-1} - y^{n-1} \geq (x - y)^{n-1}$ (for $x \geq y \geq 0$), a lower bound of the Minkowski deficit follows (cf. [26, 27]).

Proposition 1 *Let K, L be convex bodies in the Euclidean space \mathbb{R}^n , then*

$$\Delta_n(K, L) \geq (V_1(K, L)^{1/(n-1)} - r(K, L)V(L)^{1/(n-1)})^{n(n-1)}, \quad (15)$$

where the inequality holds as an equality if and only if K and L are homothetic.

The following Bonnesen-style Minkowski inequality is stronger than (15) for $n = 3$.

Theorem 1 *Let K, L be convex bodies in the Euclidean space \mathbb{R}^3 , then*

$$\Delta_3(K, L) \geq (V_1(K, L)^{1/2} - r(K, L)V(L)^{1/2})^6$$

$$+ 2r(K, L)^3 V(L)^{3/2} (V_1(K, L)^{1/2} - r(K, L)V(L)^{1/2})^3, \quad (16)$$

with equality if and only if K is homothetic to L .

Proof Since $V(K) \geq r(K, L)^3 V(L)$ and by $x^3 - y^3 \geq (x - y)^3$ (for $x \geq y \geq 0$), we have

$$\begin{aligned} V_1(K, L)^{3/2} + V(K) V(L)^{1/2} &\geq V_1(K, L)^{3/2} + r(K, L)^3 V(L)^{3/2} \\ &= V_1(K, L)^{3/2} - (r(K, L) V(L)^{1/2})^3 + 2r(K, L)^3 V(L)^{3/2} \\ &\geq (V_1(K, L)^{1/2} - r(K, L) V(L)^{1/2})^3 + 2r(K, L)^3 V(L)^{3/2}. \end{aligned}$$

Note that (14) can be rewritten as

$$V_1(K, L)^{3/2} - V(K) V(L)^{1/2} \geq (V_1(K, L)^{1/2} - r(K, L) V(L)^{1/2})^3.$$

Multiplying by $V_1(K, L)^{3/2} + V(K) V(L)^{1/2}$ on both sides, we have

$$V_1(K, L)^3 - V(K)^2 V(L) \geq (V_1(K, L)^{1/2} - r(K, L) V(L)^{1/2})^3 (V_1(K, L)^{3/2} + V(K) V(L)^{1/2}).$$

By these inequalities, we complete the proof of the theorem. \square

Let L be the unit ball and notice $S(K) = 3V_1(K, B)$ in (16), we obtain the following Bonnesen-style isoperimetric inequality that strengthens Dinghas's inequality (4) for $n = 3$.

Corollary 1 *Let K be a convex body in \mathbb{R}^3 and r be the in-radius of K , then*

$$\Delta_3(K) \geq (S(K)^{1/2} - (4\pi)^{1/2} r)^6 + 16\pi^{3/2} r^3 (S(K)^{1/2} - (4\pi)^{1/2} r)^3, \quad (17)$$

with equality if and only if K is a ball.

For $n \geq 4$, we obtain a stronger Bonnesen-style Minkowski inequality as follows.

Theorem 2 *Let K, L be convex bodies in the Euclidean space \mathbb{R}^n ($n \geq 4$), then*

$$\begin{aligned} \Delta_n(K, L) &\geq (V_1(K, L)^{1/(n-1)} - r(K, L) V(L)^{1/(n-1)})^{n(n-1)} \\ &\quad + 2(r(K, L) V(L)^{1/(n-1)})^{n(n-2)} (V_1(K, L)^{1/(n-1)} - r(K, L) V(L)^{1/(n-1)})^n \\ &\quad + (V_1(K, L) V(L))^{n/(n-1)} r(K, L)^n (V_1(K, L)^{1/(n-1)} - r(K, L) V(L)^{1/(n-1)})^{n(n-3)}, \end{aligned}$$

with equality if and only if K is homothetic to L .

Proof Let $p = V_1(K, L)^{n/(n-1)}$ and $q = V(L)^{n/(n-1)} r(K, L)^n$, then $p \geq q$.

$$\begin{aligned} &\sum_{i=2}^n (V_1(K, L)^{n(n-i)/(n-1)} (V(K)^{n-1} V(L))^{(i-2)/(n-1)}) \\ &\geq \sum_{i=2}^n (V_1(K, L)^{n(n-i)/(n-1)} (V(L)^{1/(n-1)} r(K, L))^n)^{n(i-2)} \\ &= (p^{n-2} - q^{n-2}) + pq(p^{n-4} + p^{n-5}q + \cdots + p^{n-i}q^{i-4} + \cdots + q^{n-4}) + 2q^{n-2} \\ &= (p^{n-2} - q^{n-2}) + pq \cdot \frac{p^{n-3} - q^{n-3}}{p - q} + 2q^{n-2} \end{aligned}$$

$$\begin{aligned} &\geq (p-q)^{n-2} + pq(p-q)^{n-4} + 2q^{n-2} \\ &\geq (p^{1/n} - q^{1/n})^{n(n-2)} + pq(p^{1/n} - q^{1/n})^{n(n-4)} + 2q^{n-2}. \end{aligned}$$

That is,

$$\begin{aligned} &\sum_{i=2}^n (V_1(K, L)^{n(n-i)/(n-1)} (V(K)^{n-1} V(L))^{(i-2)/(n-1)}) \\ &\geq (V_1(K, L)^{1/(n-1)} - V(L)^{1/(n-1)} r(K, L))^{n(n-2)} + 2(V(L)^{1/(n-1)} r(K, L))^{n(n-2)} \\ &\quad + (V_1(K, L) V(L))^{n/(n-1)} r(K, L)^n (V_1(K, L)^{1/(n-1)} - V(L)^{1/(n-1)} r(K, L))^{n(n-4)}. \end{aligned}$$

Multiplying by $\sum_{i=2}^n (V_1(K, L)^{n(n-i)/(n-1)} (V(K)^{n-1} V(L))^{(i-2)/(n-1)})$ both sides of (14) and via the formula

$$a^{n-1} - b^{n-1} = (a-b)(a^{n-2} + a^{n-3}b + \cdots + a^{n-i}b^{i-2} + \cdots + ab^{n-3} + b^{n-2}),$$

we obtain

$$\begin{aligned} V_1(K, L)^n - V(K)^{n-1} V(L) &\geq (V_1(K, L)^{1/(n-1)} - r(K, L) V(L)^{1/(n-1)})^n \\ &\quad \times \left(\sum_{i=2}^n (V_1(K, L)^{n(n-i)/(n-1)} (V(K)^{n-1} V(L))^{(i-2)/(n-1)}) \right). \end{aligned}$$

We complete the proof of Theorem 2. \square

Let L be the unit ball and by $S(K) = nV_1(K, B)$ in Theorem 2; we obtain the following stronger Bonnesen-style isoperimetric inequality than Dinghas's inequality (4) for $n \geq 4$.

Corollary 2 *Let K be a convex body in \mathbb{R}^n ($n \geq 4$) and r be the in-radius of K , then*

$$\begin{aligned} \Delta_n(K) &\geq (S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)} r)^{n(n-1)} \\ &\quad + 2((n\omega_n)^{1/(n-1)} r)^{n(n-2)} (S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)} r)^n \\ &\quad + (n\omega_n S(K))^{n/(n-1)} r^n (S(K)^{1/(n-1)} - (n\omega_n)^{1/(n-1)} r)^{n(n-3)}, \end{aligned}$$

with equality if and only if K is a ball.

4 Bonnesen-style Minkowski inequalities associated with the mean width

In this section, we derive some Bonnesen-style Minkowski inequalities associated with the mean width.

Lemma 2 *Let K, L be convex bodies in \mathbb{R}^n , then*

$$\frac{V_0(K, L)}{V_1(K, L)} \leq \left(\frac{V_0(K, L)}{V_n(K, L)} \right)^{1/n} \leq \frac{V_{n-1}(K, L)}{V_n(K, L)} \leq \frac{V_1(K, L)^{n-1}}{V(K)^{n-2} V(L)}, \quad (18)$$

with equality if and only if K and L are homothetic.

Proof By inequality (12), we have

$$\frac{V_{n-1}(K, L)}{V(L)} \leq \frac{V_1(K, L)^{n-1}}{V(K)^{n-2} V(L)}.$$

By the Aleksandrov-Fenchel inequality (10) we have

$$\frac{V_0(K, L)}{V_1(K, L)} \leq \frac{V_1(K, L)}{V_2(K, L)} \leq \cdots \leq \frac{V_i(K, L)}{V_{i+1}(K, L)} \leq \cdots \leq \frac{V_{n-1}(K, L)}{V_n(K, L)}.$$

Therefore

$$\frac{V_0(K, L)}{V_1(K, L)} \leq \left(\frac{V_0(K, L)}{V_n(K, L)} \right)^{1/n} \leq \frac{V_{n-1}(K, L)}{V_n(K, L)}. \quad \square$$

Theorem 3 Let K, L be convex bodies in \mathbb{R}^n , then

$$\Delta_n(K, L) \geq V(K)^{n-2} V(L) V_1(K, L) \left(\frac{V_{n-1}(K, L)}{V_n(K, L)} - \left(\frac{V_0(K, L)}{V_n(K, L)} \right)^{1/n} \right), \quad (19)$$

with equality if and only if K and L are homothetic.

Proof Via (18), we have

$$\frac{V_1(K, L)^{n-1}}{V(K)^{n-2} V(L)} - \frac{V_0(K, L)}{V_1(K, L)} \geq \frac{V_{n-1}(K, L)}{V_n(K, L)} - \left(\frac{V_0(K, L)}{V_n(K, L)} \right)^{1/n}.$$

That is

$$V_1(K, L)^n - V(K)^{n-1} V(L) \geq V(K)^{n-2} V(L) V_1(K, L) \left(\frac{V_{n-1}(K, L)}{V_n(K, L)} - \left(\frac{V_0(K, L)}{V_n(K, L)} \right)^{1/n} \right). \quad \square$$

The following Bonnesen-style inequality is a direct consequence of Theorem 3.

Theorem 4 Let K be a convex body in \mathbb{R}^n , then

$$\Delta_n(K) \geq n^{n-1} \omega_n S(K) V(K)^{n-2} \left(\frac{M(K)}{2} - \left(\frac{V(K)}{\omega_n} \right)^{1/n} \right),$$

with equality if and only if K is a ball.

Lemma 3 Let K, L be convex bodies in \mathbb{R}^n , then

$$\begin{aligned} & \frac{V_1(K, L)^{n-1} - \sqrt{V_1(K, L)^{n-2} (V_1(K, L)^n - V(K)^{n-1} V(L))}}{V(K)^{n-2} V(L)} \\ & \leq \frac{V_0(K, L)}{V_1(K, L)} \leq \frac{V_{n-1}(K, L)}{V_n(K, L)} \leq \frac{V_1(K, L)^{n-1}}{V(K)^{n-2} V(L)}. \end{aligned} \quad (20)$$

Proof The Minkowski inequality (5) gives

$$\frac{V_1(K, L)^{n-1} - \sqrt{V_1(K, L)^{n-2} (V_1(K, L)^n - V(K)^{n-1} V(L))}}{V(K)^{n-2} V(L)} \leq \frac{V_0(K, L)}{V_1(K, L)}.$$

The above inequality together with (18) leads to Lemma 3. □

We are now in a position to prove the following Bonnesen-style Minkowski inequality.

Theorem 5 *Let K, L be convex bodies in \mathbb{R}^n , then*

$$\Delta_n(K, L) \geq \frac{V(K)^{2n-4} V(L)^2}{V_1(K, L)^{n-2}} \left(\frac{V_{n-1}(K, L)}{V_n(K, L)} - \frac{V_0(K, L)}{V_1(K, L)} \right)^2, \quad (21)$$

with equality if and only if K and L are homothetic.

Proof From (20) we have

$$\frac{\sqrt{V_1(K, L)^{n-2} (V_1(K, L)^n - V(K)^{n-1} V(L))}}{V(K)^{n-2} V(L)} \geq \frac{V_{n-1}(K, L)}{V_n(K, L)} - \frac{V_0(K, L)}{V_1(K, L)}. \quad \square$$

The following Bonnesen-style inequality is a direct consequence of Theorem 5 when L is the unit ball.

Theorem 6 *Let K be a convex body in \mathbb{R}^n , then*

$$S(K)^n - n^n \omega_n V(K)^{n-1} \geq \frac{n^{2n-2} \omega_n^2 V^{2n-4}}{S^{n-2}} \left(\frac{M(K)}{2} - \frac{nV(K)}{S(K)} \right)^2,$$

with equality if and only if K is a ball.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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